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Evolution and impedance operators of spherically symmetric bianisotropic media

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Abstract

An operator method, which was developed earlier for bianisotropic layered media with plane and cylindrical interfaces, is generalized to solve spherically symmetric problems: determination of eigenmodes of spherical waveguides, investigation of surface waves, calculation of light scattering cross-section by multi-layer spherical particles. Analytical results are applied to obtain the modes of the negative-refractive-index waveguide, to determine dispersion curves of the waves at the spherical interface between isotropic and bianisotropic media, to reveal the regularities of electromagnetic wave scattering by spherical dielectric particles.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In electrodynamics there are a number of problems in obtaining the exact analytical solution of which a certain symmetry (plane, cylindrical, spherical) is required. Some such problems are obtaining waveguide eigenwaves, the calculation of scattering of electromagnetic waves by particles, the determination of multipole radiation, the study of light beam propagation, etc. In the present paper we are interested in the solution of some of the problems mentioned above which possess spherical symmetry.

Spherical waveguides have been studied for a long time [1] and applied to propagation of radio waves round the Earth. In the paper [2] electromagnetic waves in a spherically layered anisotropic dissipative medium were investigated by the normal wave method, which is based on the spectral theory of linear non-self-conjugate operators. The computation of the wave numbers of waveguide normal waves can be realized by the impedance recalculation method [3, 4].

Theoretically the electromagnetic wave scattering by spherical particles [5, 6] is studied numerically [7] or using the spherical vector wavefunctions [8, 9]. Such functions are the solutions of the vector Helmholtz equation. They depend on each spherical coordinate r, θ, φ. The scattering problem lies in determining coefficients of the spherical vector wavefunctions from boundary conditions.

The operator method was applied earlier for the study of the reflection and guiding in multi-layer bianisotropic structures with plane [10, 11] or cylindrical [12, 13] symmetry. In the framework of the operator approach one should separate the variables in Maxwell's equations reducing the Maxwell equations to the system of equations of first order for tangential components of the electric and magnetic fields [14]. The tangential components are situated in the plane tangent to the interface between two media. Evolution operator (characteristic matrix, transfer matrix) and impedance tensors of partial waves are the solutions of the system of equations of first order. Using evolution and impedance operators one can determine reflection and transmission coefficients of electromagnetic waves propagating in layered media, as well as dispersion equations and polarizations of the waveguide eigenmodes. It should be noted that spatial evolution of cylindrical beams in complex media can be also investigated by means of operator approach [15, 16].

In this investigation the operator method is generalized to the case of spherically symmetric bianisotropic media. The set of differential equations of first order for spherical waves is derived in section 2 and solved in section 3. Section 4 is devoted to the ascertainment of the form of the evolution operators and impedance tensors for isotropic and bianisotropic spherically symmetric layers. In section 5 we apply operator technique to study eigenmodes of multi-layer spherical guides, surface waves at spherical interface, light scattering by the multi-layer spherical particles.

2. Ordinary differential equations of the first order for spherical waves in bianisotropic media

Classical electromagnetic waves in bianisotropic media

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} + \alpha \boldsymbol{H} \qquad \boldsymbol{B} = \kappa \boldsymbol{E} + \mu \boldsymbol{H} \tag{1}$$

satisfy the Maxwell equations

$$\begin{aligned} \left(\boldsymbol{e}_r^{\times} \frac{\partial}{\partial r} + \boldsymbol{e}_\theta^{\times} \frac{1}{r} \frac{\partial}{\partial \theta} + \boldsymbol{e}_\varphi^{\times} \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \right) \boldsymbol{H}(\boldsymbol{r}, t) &= \frac{1}{c} \frac{\partial \boldsymbol{D}(\boldsymbol{r}, t)}{\partial t} \\ \left(\boldsymbol{e}_r^{\times} \frac{\partial}{\partial r} + \boldsymbol{e}_\theta^{\times} \frac{1}{r} \frac{\partial}{\partial \theta} + \boldsymbol{e}_\varphi^{\times} \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \right) \boldsymbol{E}(\boldsymbol{r}, t) &= -\frac{1}{c} \frac{\partial \boldsymbol{B}(\boldsymbol{r}, t)}{\partial t}, \end{aligned} \tag{2}$$

where (r, θ, φ) are the spherical coordinates; $\boldsymbol{e}_r(\theta, \varphi)$, $\boldsymbol{e}_\theta(\theta, \varphi)$, $\boldsymbol{e}_\varphi(\varphi)$ are the base vectors of the spherical coordinates; $\boldsymbol{e}_r^{\times}$ is the tensor dual to the vector $\boldsymbol{e}_r$ [17]; $\boldsymbol{H}, \boldsymbol{E}, \boldsymbol{B}$ and $\boldsymbol{D}$ are the strengths and inductions of the magnetic and electric fields. Suppose that dielectric permittivity ε and magnetic permeability μ tensors, as well as gyration pseudotensors α, κ can be written as

$$\xi(r) = \xi_1(r) \boldsymbol{e}_r \otimes \boldsymbol{e}_r + \xi_2(r) I + \mathrm{i}\chi_\xi(r) \boldsymbol{e}_r^{\times}, \tag{3}$$

where ξ corresponds to one of the tensors ε, μ, α, κ; $\boldsymbol{e}_r \otimes \boldsymbol{e}_r$ is the elementary dyad; $I = 1 - \boldsymbol{e}_r \otimes \boldsymbol{e}_r = -\boldsymbol{e}_r^{\times 2}$ is the projection operator onto the plane tangent to the spherical surface (the plane normal to the vector $\boldsymbol{e}_r$). In Maxwell's equations (2) one can easily separate the angle φ and time t from the rest of coordinates as

$$\begin{pmatrix} \boldsymbol{H}(\boldsymbol{r}, t) \\ \boldsymbol{E}(\boldsymbol{r}, t) \end{pmatrix} = \mathrm{e}^{-\mathrm{i}\omega t} \sum_{m \in Z} \mathrm{e}^{\mathrm{i}m\varphi} \begin{pmatrix} \boldsymbol{H}(r, \theta, m) \\ \boldsymbol{E}(r, \theta, m) \end{pmatrix}, \tag{4}$$

where ω is the wave circular frequency. Taking into account the φ-differentiation of the fields

$$\frac{\partial \boldsymbol{H}(\boldsymbol{r},t)}{\partial \varphi} = \mathrm{e}^{\mathrm{i}m\varphi}\left(\mathrm{i}m - \sin\theta\boldsymbol{e}_\theta^\times + \cos\theta\boldsymbol{e}_r^\times\right)\boldsymbol{H}(r,\theta) \tag{5}$$

the Maxwell equations (2) can be reduced to the following form:

$$\begin{aligned}
&\left(\boldsymbol{e}_r^\times\frac{\partial}{\partial r} + \boldsymbol{e}_\theta^\times\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\mathrm{i}m}{r\sin\theta}\boldsymbol{e}_\varphi^\times - \frac{1}{r}\boldsymbol{e}_\theta\otimes\boldsymbol{e}_\varphi + \frac{\cot\theta}{r}\boldsymbol{e}_r\otimes\boldsymbol{e}_\varphi\right)\boldsymbol{H}(r,\theta)\\
&\qquad = -\mathrm{i}k(\varepsilon\boldsymbol{E}(r,\theta) + \alpha\boldsymbol{H}(r,\theta))\\
&\left(\boldsymbol{e}_r^\times\frac{\partial}{\partial r} + \boldsymbol{e}_\theta^\times\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\mathrm{i}m}{r\sin\theta}\boldsymbol{e}_\varphi^\times - \frac{1}{r}\boldsymbol{e}_\theta\otimes\boldsymbol{e}_\varphi + \frac{\cot\theta}{r}\boldsymbol{e}_r\otimes\boldsymbol{e}_\varphi\right)\boldsymbol{E}(r,\theta)\\
&\qquad = \mathrm{i}k(\kappa\boldsymbol{E}(r,\theta) + \mu\boldsymbol{H}(r,\theta)),
\end{aligned} \tag{6}$$

where $k = \omega/c$ is the vacuum wave number.

The θ-dependence of the field strengths cannot be expressed by means of a scalar function. One should use tensor $F(\theta)$:

$$\begin{pmatrix}\boldsymbol{H}(r,\theta)\\ \boldsymbol{E}(r,\theta)\end{pmatrix} = \begin{pmatrix}F(\theta)\boldsymbol{H}(r)\\ F(\theta)\boldsymbol{E}(r)\end{pmatrix}. \tag{7}$$

In order to separate variables r and θ according to formula (7) the tensor F should satisfy the commutation relations

$$[\xi(r), F(\theta)] = \left[\boldsymbol{e}_r^\times, F(\theta)\right] = 0 \tag{8}$$

as well as the expression

$$F(\theta)^{-1}\left(\boldsymbol{e}_\theta^\times\frac{\partial}{\partial\theta} + \frac{\mathrm{i}m}{\sin\theta}\boldsymbol{e}_\varphi^\times - \boldsymbol{e}_\theta\otimes\boldsymbol{e}_\varphi + \cot\theta\boldsymbol{e}_r\otimes\boldsymbol{e}_\varphi\right)F(\theta) = G. \tag{9}$$

The components of tensor G in spherical coordinates do not depend on the angle θ.

From conditions (8) it follows that the tensor F is of the form

$$F(\theta) = f_1(\theta)\boldsymbol{e}_r\otimes\boldsymbol{e}_r + f_2(\theta)I + f_3(\theta)\boldsymbol{e}_r^\times. \tag{10}$$

Complex functions $f_1(\theta)$, $f_2(\theta)$, $f_3(\theta)$ can be found from equation (9). Tensor G equals

$$G = \alpha_1\boldsymbol{e}_r\otimes\boldsymbol{e}_\varphi + \alpha_2\boldsymbol{e}_r\otimes\boldsymbol{e}_\theta - \alpha_3\boldsymbol{e}_\varphi\otimes\boldsymbol{e}_r - \alpha_4\boldsymbol{e}_\theta\otimes\boldsymbol{e}_r + \boldsymbol{e}_r^\times \tag{11}$$

while functions $f_1(\theta)$, $f_2(\theta)$, $f_3(\theta)$ satisfy the four differential equations:

$$\begin{aligned}
&\frac{\mathrm{d}f_2}{\mathrm{d}\theta} + \frac{\mathrm{i}m}{\sin\theta}f_3 + \cot\theta f_2 = \alpha_1 f_1\\
&\frac{\mathrm{d}f_3}{\mathrm{d}\theta} - \frac{\mathrm{i}m}{\sin\theta}f_2 + \cot\theta f_3 = \alpha_2 f_1\\
&\frac{\mathrm{d}f_1}{\mathrm{d}\theta}f_2 + \frac{\mathrm{i}m}{\sin\theta}f_1 f_3 = \alpha_3\left(f_2^2 + f_3^2\right)\\
&\frac{\mathrm{d}f_1}{\mathrm{d}\theta}f_3 - \frac{\mathrm{i}m}{\sin\theta}f_1 f_2 = \alpha_4\left(f_2^2 + f_3^2\right),
\end{aligned} \tag{12}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are constant coefficients. Let us assume that the following conditions hold true: $f_1 = f_1^*$, $f_2 = \mathrm{i}\tilde{f}_2$, $\tilde{f}_2 = \tilde{f}_2{}^*$, $f_3 = f_3^*$, $\alpha_1 = 0$, $\alpha_2 = l(l+1)$, $\alpha_3 = 0$, $\alpha_4 = -1$, where the symbol * denotes the complex conjugate. Then the equations (12) can be rewritten as the set of equations for real functions f_1, $\tilde{f}_2$, f_3:

$$\frac{\mathrm{d}\tilde{f}_2}{\mathrm{d}\theta} + \frac{m}{\sin\theta}f_3 + \cot\theta\tilde{f}_2 = 0 \tag{13}$$

$$\frac{\mathrm{d}f_3}{\mathrm{d}\theta} + \frac{m}{\sin\theta}\tilde{f}_2 + \cot\theta f_3 = l(l+1)f_1 \tag{14}$$

$$\frac{\mathrm{d}f_1}{\mathrm{d}\theta}\tilde{f}_2 + \frac{m}{\sin\theta}f_1 f_3 = 0 \tag{15}$$

$$\frac{\mathrm{d}f_1}{\mathrm{d}\theta}f_3 + \frac{m}{\sin\theta}f_1\tilde{f}_2 = \tilde{f}_2^2 - f_3^2. \tag{16}$$

From equations (15), (16) it follows the link between functions $\tilde{f}_2$, f_3 and f_1:

$$\tilde{f}_2(\theta) = \frac{mf_1}{\sin\theta}, \qquad f_3(\theta) = -\frac{\mathrm{d}f_1}{\mathrm{d}\theta}. \tag{17}$$

Then equation (13) becomes identity. From (14) we obtain

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}f_1}{\mathrm{d}\theta}\right) + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)f_1 = 0. \tag{18}$$

The solutions of this equation are the associated Legendre functions:

$$f_1(\theta) = P_l^{|m|}(\cos\theta), \qquad \tilde{f}_2(\theta) = \frac{m}{\sin\theta}P_l^{|m|}(\cos\theta), \qquad f_3(\theta) = -\frac{\mathrm{d}P_l^{|m|}(\cos\theta)}{\mathrm{d}\theta}. \tag{19}$$

The order of Legendre polynomials is always positive, even for negative values m. So, the tensor $F(\theta)$ is equal to

$$\begin{aligned} F_l^m(\theta) &= P_l^{|m|}(\cos\theta)\boldsymbol{e}_r\otimes\boldsymbol{e}_r + \frac{\mathrm{i}m}{\sin\theta}P_l^{|m|}(\cos\theta)\boldsymbol{I} - \frac{\mathrm{d}P_l^{|m|}(\cos\theta)}{\mathrm{d}\theta}\boldsymbol{e}_r^{\times} \\ &\equiv \left(\boldsymbol{e}_r\otimes\boldsymbol{e}_r + \frac{\mathrm{i}m}{\sin\theta}\boldsymbol{I}\right)P_l^{|m|} - \frac{l(l-|m|+1)}{2l+1}\boldsymbol{e}_r^{\times}\frac{P_{l+1}^{|m|}}{\sin\theta} + \frac{(l+|m|)(l+1)}{2l+1}\boldsymbol{e}_r^{\times}\frac{P_{l-1}^{|m|}}{\sin\theta}. \end{aligned} \tag{20}$$

Field strengths $\boldsymbol{H}(r)$ and $\boldsymbol{E}(r)$ introduced in equation (7) satisfy equations

$$\begin{aligned} &\left(\boldsymbol{e}_r^{\times}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r}\boldsymbol{e}_r^{\times} + \frac{l(l+1)}{r}\boldsymbol{e}_r\otimes\boldsymbol{e}_\theta + \frac{1}{r}\boldsymbol{e}_\theta\otimes\boldsymbol{e}_r\right)\boldsymbol{H}(r) = -\mathrm{i}k(\varepsilon\boldsymbol{E}(r) + \alpha\boldsymbol{H}(r)) \\ &\left(\boldsymbol{e}_r^{\times}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1}{r}\boldsymbol{e}_r^{\times} + \frac{l(l+1)}{r}\boldsymbol{e}_r\otimes\boldsymbol{e}_\theta + \frac{1}{r}\boldsymbol{e}_\theta\otimes\boldsymbol{e}_r\right)\boldsymbol{E}(r) = \mathrm{i}k(\kappa\boldsymbol{E}(r) + \mu\boldsymbol{H}(r)). \end{aligned} \tag{21}$$

Vectors $\boldsymbol{H}(r)$ and $\boldsymbol{E}(r)$ can be found for each specific bianisotropic medium (3). The fields $\boldsymbol{H}(r)$ and $\boldsymbol{E}(r)$ depend on the angles θ and φ, which enter only into base vectors $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\varphi$. Field coordinates are merely functions of r.

Besides the differential equations, equations (21) contain also two algebraic equations, which allow us to exclude two components of field vectors. Eliminating H_r and E_r one can write the system of ordinary differential equations of first order for the rest tangential field components $\boldsymbol{H}_\mathrm{t} = \boldsymbol{I}\boldsymbol{H}$, $\boldsymbol{E}_\mathrm{t} = \boldsymbol{I}\boldsymbol{E}$ as

$$\frac{\mathrm{d}\boldsymbol{W}(r)}{\mathrm{d}r} = \mathrm{i}kM(r)\boldsymbol{W}(r), \tag{22}$$

where

$$\begin{aligned} &M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \boldsymbol{W} = \begin{pmatrix} \boldsymbol{H}_\mathrm{t} \\ \boldsymbol{E}_\mathrm{t} \end{pmatrix} \\ &A = \frac{\mathrm{i}}{kr}\boldsymbol{I} + \boldsymbol{e}_r^{\times}\alpha(r)\boldsymbol{I} - \frac{p\kappa_1(r)}{k^2r^2}\boldsymbol{e}_\varphi\otimes\boldsymbol{e}_\theta \qquad B = \boldsymbol{e}_r^{\times}\varepsilon(r)\boldsymbol{I} - \frac{p\varepsilon_1(r)}{k^2r^2}\boldsymbol{e}_\varphi\otimes\boldsymbol{e}_\theta \\ &C = -\boldsymbol{e}_r^{\times}\mu(r)\boldsymbol{I} + \frac{p\mu_1(r)}{k^2r^2}\boldsymbol{e}_\varphi\otimes\boldsymbol{e}_\theta \qquad D = \frac{\mathrm{i}}{kr}\boldsymbol{I} - \boldsymbol{e}_r^{\times}\kappa(r)\boldsymbol{I} + \frac{p\alpha_1(r)}{k^2r^2}\boldsymbol{e}_\varphi\otimes\boldsymbol{e}_\theta \\ &p(r) = l(l+1)/(\varepsilon_1(r)\mu_1(r) - \alpha_1(r)\kappa_1(r)). \end{aligned} \tag{23}$$

Total fields can be found using the matrix V:

$$\begin{pmatrix} \boldsymbol{H} \\ \boldsymbol{E} \end{pmatrix} = V \begin{pmatrix} \boldsymbol{H}_{\mathrm{t}} \\ \boldsymbol{E}_{\mathrm{t}} \end{pmatrix}$$
$$V = \begin{pmatrix} I - \frac{\mathrm{i}p\kappa_1(r)}{kr}\boldsymbol{e}_r \otimes \boldsymbol{e}_\theta & -\frac{\mathrm{i}p\varepsilon_1(r)}{kr}\boldsymbol{e}_r \otimes \boldsymbol{e}_\theta \\ \frac{\mathrm{i}p\mu_1(r)}{kr}\boldsymbol{e}_r \otimes \boldsymbol{e}_\theta & I + \frac{\mathrm{i}p\alpha_1(r)}{kr}\boldsymbol{e}_r \otimes \boldsymbol{e}_\theta \end{pmatrix}. \tag{24}$$

Equations of the form (22) were used earlier in papers [10, 12] for layered media with plane or cylindrical interfaces. The fact, that we can reduce the Maxwell equations to the system of differential equations of first order for spherically symmetric problems, plays an important part, because we can apply the formulae derived earlier for electromagnetic wave reflection and guiding to spherically layered media. The solution of equation (22) can be expressed by means of product integral for an arbitrary inhomogeneous bianisotropic medium. In the following section we will obtain the analytical solutions for homogeneous media.

3. Radial solutions for spherical waves in homogeneous bianisotropic media

We define homogeneous bianisotropic media as media with constant values ξ_1, ξ_2, χ_ξ. In fact, such homogeneous media are inhomogeneous, because the base vector $\boldsymbol{e}_r$ depends on the angles θ and φ. Nevertheless, consideration of such defined homogeneous media possessing the spherical symmetry allows us to find the analytical solutions of the system of differential equations of first order.

The solutions of equation (22) can be determined by the same way as in the paper [12] for cylindrical waves. Matrix M can be expanded in $1/r$ series as

$$M = M^{(0)} + \frac{1}{r}M^{(1)} + \frac{1}{r^2}M^{(2)}. \tag{25}$$

Constant matrices $M^{(0)}$, $M^{(1)}$, $M^{(2)}$ are equal to

$$M^{(0)} = \begin{pmatrix} \boldsymbol{e}_r^\times \alpha I & \boldsymbol{e}_r^\times \varepsilon I \\ -\boldsymbol{e}_r^\times \mu I & -\boldsymbol{e}_r^\times \kappa I \end{pmatrix} \qquad M^{(1)} = \frac{\mathrm{i}}{k}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \frac{\mathrm{i}}{k}E$$
$$M^{(2)} = -\frac{p}{k^2}\begin{pmatrix} \kappa_2 \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta & \varepsilon_2 \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta \\ -\mu_2 \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta & -\alpha_2 \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta \end{pmatrix}. \tag{26}$$

Let us present tangential field vector $\boldsymbol{W}$ and operator matrix M as base vector decomposition:

$$\boldsymbol{W} = \vec{w}_\varphi \boldsymbol{e}_\varphi + \vec{w}_\theta \boldsymbol{e}_\theta \qquad \vec{w}_\varphi = \boldsymbol{e}_\varphi \boldsymbol{W} = \begin{pmatrix} \boldsymbol{e}_\varphi \boldsymbol{H} \\ \boldsymbol{e}_\varphi \boldsymbol{E} \end{pmatrix} = \begin{pmatrix} H_\varphi \\ E_\varphi \end{pmatrix} \qquad \vec{w}_\theta = \begin{pmatrix} H_\theta \\ E_\theta \end{pmatrix}$$
$$M = M_{\theta\theta}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + M_{\theta\varphi}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\varphi + M_{\varphi\theta}\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta + M_{\varphi\varphi}\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\varphi$$
$$M_{\theta\theta} = \boldsymbol{e}_\theta M \boldsymbol{e}_\theta = \begin{pmatrix} \boldsymbol{e}_\theta A \boldsymbol{e}_\theta & \boldsymbol{e}_\theta B \boldsymbol{e}_\theta \\ \boldsymbol{e}_\theta C \boldsymbol{e}_\theta & \boldsymbol{e}_\theta D \boldsymbol{e}_\theta \end{pmatrix} = \begin{pmatrix} A_{\theta\theta} & B_{\theta\theta} \\ C_{\theta\theta} & D_{\theta\theta} \end{pmatrix}$$
$$M_{\theta\varphi} = \boldsymbol{e}_\theta M \boldsymbol{e}_\varphi \qquad M_{\varphi\theta} = \boldsymbol{e}_\varphi M \boldsymbol{e}_\theta \qquad M_{\varphi\varphi} = \boldsymbol{e}_\varphi M \boldsymbol{e}_\varphi.$$

Block matrices $M^{(0)}$, $M^{(1)}$, $M^{(2)}$ equal

$$M^{(0)} = M_{\theta\theta}^{(0)} I + M_{\theta\varphi}^{(0)} \boldsymbol{e}_r^\times = \mathrm{i}\begin{pmatrix} -\chi_\alpha & -\chi_\varepsilon \\ \chi_\mu & \chi_\kappa \end{pmatrix} I - \begin{pmatrix} \alpha_2 & \varepsilon_2 \\ -\mu_2 & -\kappa_2 \end{pmatrix} \boldsymbol{e}_r^\times$$
$$M^{(1)} = M_{\theta\theta}^{(1)} I = \frac{\mathrm{i}}{k}\hat{1} I \qquad M^{(2)} = M_{\varphi\theta}^{(2)} \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta = -\frac{p}{k^2}\begin{pmatrix} \kappa_2 & \varepsilon_2 \\ -\mu_2 & -\alpha_2 \end{pmatrix} \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta, \tag{27}$$

where $\hat{1}$ is the 2×2 unit matrix.

From equation (22) it follows the ordinary differential equation of the second order for the two-dimensional vector $\vec{w}_\theta$:

$$\vec{w}_\theta'' + \left(P + \frac{2}{r}\hat{1}\right)\vec{w}_\theta' + \left(Q + \frac{1}{r}P - \frac{\nu^2}{r^2}\hat{1}\right)\vec{w}_\theta = 0, \tag{28}$$

where

$$\begin{aligned} P &= -\mathrm{i}k\left(M_{\theta\theta}^{(0)} + M_{\theta\varphi}^{(0)} M_{\theta\theta}^{(0)} M_{\theta\varphi}^{(0)-1}\right) \\ Q &= -k^2 M_{\theta\varphi}^{(0)}\left(M_{\theta\varphi}^{(0)} + M_{\theta\theta}^{(0)} M_{\theta\varphi}^{(0)-1} M_{\theta\theta}^{(0)}\right) \\ \nu^2 &= l(l+1)\frac{\varepsilon_2\mu_2 - \alpha_2\kappa_2}{\varepsilon_1\mu_1 - \alpha_1\kappa_1}. \end{aligned} \tag{29}$$

If matrix Q commutes with P, then one should search the solution of equation (28) as follows:

$$\vec{w}_\theta(r) = \frac{1}{\sqrt{r}}\exp\left(-\frac{1}{2}Pr\right)\vec{y}(r). \tag{30}$$

It can be found by the direct computation the condition for commutation of two matrices P and Q: $c_1 P + c_2 Q = c_3\hat{1}$, where c_1, c_2, c_3 are constants. So, the commutation is possible in the following cases: (i) $P = 0$ $(c_2 = c_3 = 0)$ or $Q = 0$ $(c_1 = c_3 = 0)$; (ii) $P \sim \hat{1}$ $(c_2 = 0)$ or $Q \sim \hat{1}$ $(c_1 = 0)$; (iii) $P \sim Q$ $(c_3 = 0)$. In further examples (see section 4) matrix Q commutes with P.

Vector $\vec{y}$ satisfies the Bessel equation

$$\vec{y}'' + \frac{1}{r}\vec{y}' + \left(Q - \frac{1}{4}P^2 - \frac{\nu^2 + 1/4}{r^2}\hat{1}\right)\vec{y} = 0. \tag{31}$$

The solution of equation (31) is the Bessel function with matrix argument. Therefore, the solutions $\vec{w}_\theta$ are equal to

$$\begin{aligned} &\vec{w}_\theta = T_1(c_1\vec{a}_1 + c_1\vec{a}_2) + T_2(c_3\vec{a}_3 + c_4\vec{a}_4) \\ &T_1 = \sqrt{\frac{\pi}{2r}}\exp\left(-\frac{1}{2}Pr\right) J_{\sqrt{\nu^2+1/4}}\left(\sqrt{Q - \frac{1}{4}P^2}, r\right) \equiv \exp\left(-\frac{1}{2}Pr\right) j_{s_1}\left(\sqrt{Q - \frac{1}{4}P^2 r}\right) \\ &T_2 = \sqrt{\frac{\pi}{2r}}\exp\left(-\frac{1}{2}Pr\right) J_{-\sqrt{\nu^2+1/4}}\left(\sqrt{Q - \frac{1}{4}P^2 r}\right) \equiv \exp\left(-\frac{1}{2}Pr\right) j_{s_2}\left(\sqrt{Q - \frac{1}{4}P^2 r}\right) \\ &s_1 = \sqrt{\nu^2 + 1/4} - 1/2 \qquad s_2 = -\sqrt{\nu^2 + 1/4} - 1/2, \end{aligned} \tag{32}$$

where j_s is the spherical Bessel function of the order s; c_1, c_2, c_3, c_4 are constants of integration; $\vec{a}_1, \vec{a}_2$ and $\vec{a}_3, \vec{a}_4$ are two couples of arbitrary noncollinear vectors. These vectors can be chosen as unit two-dimensional vectors $\vec{e}_1 = (1, 0)^T$, $\vec{e}_2 = (0, 1)^T$ or as eigenvectors of the matrix $Q - P^2/4$. To calculate the function of matrix one can use the spectral decomposition of both matrix P and matrix $Q - P^2/4$. The solutions of equation (31) are not necessarily Bessel functions, but are Hankel functions and modified Bessel functions with matrix argument $\sqrt{P^2/4 - Q r}$, too.

Introducing the differential operator

$$\hat{Z} = M_{\theta\varphi}^{(0)-1}\left(\frac{1}{\mathrm{i}k}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\mathrm{i}}{kr} - M_{\theta\theta}^{(0)}\right)$$

we can obtain the tangential magnetic and electric field strengths $\boldsymbol{W}$ as

$$\boldsymbol{W} = \vec{w}_\theta(r)e_\theta + \hat{Z}\vec{w}_\theta(r)e_\varphi. \tag{33}$$

Four constants c_j, $j = 1, 2, 3, 4$ can be included into the vectors of the three-dimensional space $\boldsymbol{c}_1 = c_1\boldsymbol{e}_\theta + c_2\boldsymbol{e}_\varphi$ and $\boldsymbol{c}_2 = c_3\boldsymbol{e}_\theta + c_4\boldsymbol{e}_\varphi$ as their components. Hence, the vector $\boldsymbol{W}$ takes the form

$$\boldsymbol{W} = S(r)\boldsymbol{C} \qquad S = \begin{pmatrix} \eta_1 & \eta_2 \\ \zeta_1 & \zeta_2 \end{pmatrix} \qquad \boldsymbol{C} = \begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \end{pmatrix}. \tag{34}$$

Blocks of the matrix S are expressed by means of the matrix functions T_1 and T_2:

$$\begin{aligned}
\eta_1 &= \vec{e}_1 T_1 \vec{a}_1 \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \vec{e}_1 \hat{Z} T_1 \vec{a}_1 \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta + \vec{e}_1 T_1 \vec{a}_2 \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\varphi + \vec{e}_1 \hat{Z} T_1 \vec{a}_2 \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\varphi \\
\eta_2 &= \vec{e}_1 T_2 \vec{a}_3 \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \vec{e}_1 \hat{Z} T_2 \vec{a}_3 \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta + \vec{e}_1 T_2 \vec{a}_4 \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\varphi + \vec{e}_1 \hat{Z} T_2 \vec{a}_4 \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\varphi \\
\zeta_1 &= \vec{e}_2 T_1 \vec{a}_1 \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \vec{e}_2 \hat{Z} T_1 \vec{a}_1 \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta + \vec{e}_2 T_1 \vec{a}_2 \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\varphi + \vec{e}_2 \hat{Z} T_1 \vec{a}_2 \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\varphi \\
\zeta_2 &= \vec{e}_2 T_2 \vec{a}_3 \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + \vec{e}_2 \hat{Z} T_2 \vec{a}_3 \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta + \vec{e}_2 T_2 \vec{a}_4 \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\varphi + \vec{e}_2 \hat{Z} T_2 \vec{a}_4 \boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\varphi.
\end{aligned} \tag{35}$$

From (34) it follows the expression for the evolution operator Ω_a^r, which determines the tangential field components in the point r, if the fields in the point a are known:

$$\boldsymbol{W}(r) = \Omega_a^r \boldsymbol{W}(a) \qquad \Omega_a^r = S(r)S^-(a). \tag{36}$$

Pseudoinverse matrix S^- satisfies conditions $SS^- = S^-S = E$ [14, 17], where matrix E is introduced in (26). Representing equation (34) as superposition of two waves

$$\boldsymbol{W} = \begin{pmatrix} \boldsymbol{H}_{\mathrm{t}1} \\ \boldsymbol{E}_{\mathrm{t}1} \end{pmatrix} + \begin{pmatrix} \boldsymbol{H}_{\mathrm{t}2} \\ \boldsymbol{E}_{\mathrm{t}2} \end{pmatrix} \tag{37}$$

and using the definition of the impedance tensor $\boldsymbol{E}_{\mathrm{t}i} = \Gamma_i \boldsymbol{H}_{\mathrm{t}i}$, $i = 1, 2$, it is easily to find the impedance tensors of each of two partial waves

$$\Gamma_i = \zeta_i \eta_i^-, \tag{38}$$

where $\boldsymbol{H}_{\mathrm{t}1} = \eta_1 \boldsymbol{c}_1$, $\boldsymbol{H}_{\mathrm{t}2} = \eta_2 \boldsymbol{c}_2$, $\boldsymbol{E}_{\mathrm{t}1} = \zeta_1 \boldsymbol{c}_1$, $\boldsymbol{E}_{\mathrm{t}2} = \zeta_2 \boldsymbol{c}_2$, η_i^- is the pseudoinverse tensor ($\eta_i^- \eta_i = \eta_i \eta_i^- = I$). If the initial magnetic field amplitudes of the partial waves $\boldsymbol{H}_{\mathrm{t}1}(a)$ and $\boldsymbol{H}_{\mathrm{t}2}(a)$ are known, then the waves propagate according to the relations

$$\boldsymbol{H}_{\mathrm{t}i}(r) = \eta_i(r)\eta_i^-(a)\boldsymbol{H}_{\mathrm{t}i}(a) \qquad \boldsymbol{E}_{\mathrm{t}i}(r) = \zeta_i(r)\eta_i^-(a)\boldsymbol{H}_{\mathrm{t}i}(a). \tag{39}$$

4. Examples of the radial solutions

4.1. Isotropic spherical layer

Matrices (27) can be easily calculated for an isotropic medium with scalar dielectric permittivity ε and magnetic permeability μ and $\alpha = \kappa = 0$:

$$M_{\theta\theta}^{(0)} = 0 \qquad M_{\theta\varphi}^{(0)} = \begin{pmatrix} 0 & -\varepsilon \\ \mu & 0 \end{pmatrix}. \tag{40}$$

Therefore, the quantities P, Q and ν^2 equal

$$P = 0 \qquad Q = k^2 \varepsilon\mu \hat{1} \qquad \nu^2 = l(l+1) \tag{41}$$

while the matrix solutions take the form

$$T_1 = j_l(k\sqrt{\varepsilon\mu}r)\hat{1} \qquad T_2 = j_{-l-1}(k\sqrt{\varepsilon\mu}r)\hat{1}. \tag{42}$$

One can write blocks of the matrix S using formulae (35):

$$\begin{aligned}
\eta_1 &= j_l \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta - \frac{\mathrm{i}}{\mu k r}\frac{\mathrm{d}(r j_l)}{\mathrm{d}r}\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\varphi \\
\eta_2 &= j_{-l-1} \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta - \frac{\mathrm{i}}{\mu k r}\frac{\mathrm{d}(r j_{-l-1})}{\mathrm{d}r}\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\varphi \\
\zeta_1 &= j_l \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\varphi + \frac{\mathrm{i}}{\varepsilon k r}\frac{\mathrm{d}(r j_l)}{\mathrm{d}r}\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta \\
\zeta_2 &= j_{-l-1} \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\varphi + \frac{\mathrm{i}}{\varepsilon k r}\frac{\mathrm{d}(r j_{-l-1})}{\mathrm{d}r}\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta .
\end{aligned} \tag{43}$$

Impedance tensors of a homogeneous isotropic spherical layer are equal to

$$\begin{aligned}
\Gamma_1(r) &= \frac{\mathrm{i}}{k\varepsilon}\frac{(r j_l)'}{r j_l}\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta + \mathrm{i}k\mu\frac{r j_l}{(r j_l)'}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\varphi \\
\Gamma_2(r) &= \frac{\mathrm{i}}{k\varepsilon}\frac{(r j_{-l-1})'}{r j_{-l-1}}\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta + \mathrm{i}k\mu\frac{r j_{-l-1}}{(r j_{-l-1})'}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\varphi ,
\end{aligned} \tag{44}$$

where prime denotes the r-derivative. The field radial solutions can be expressed by means of the Hankel functions, too. In this case in equations (43) and (44) the Bessel functions j_l should be replaced by the Hankel functions. The spherical Hankel function of the first (second) kind $h_l^{(1,2)}(r) = j_l(r) \pm \mathrm{i} y_l(r)$ corresponds to the divergent (converging) spherical wave, where $y_l(r) = j_{-l-1}(r)$ is the spherical Bessel function of the second kind. The quantities containing Hankel functions we will denote as the letters with tilde. For instance, the impedance tensor of the divergent wave takes the form

$$\widetilde{\Gamma}_1(r) = \frac{\mathrm{i}}{k\varepsilon}\frac{\left(r h_l^{(1)}\right)'}{r h_l^{(1)}}\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta + \mathrm{i}k\mu\frac{r h_l^{(1)}}{\left(r h_l^{(1)}\right)'}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\varphi . \tag{45}$$

4.2. Bianisotropic spherical layer

We consider a bianisotropic medium, which is characterized by the following tensor parameters: $\varepsilon = \varepsilon_1 \boldsymbol{e}_r \otimes \boldsymbol{e}_r + \varepsilon_2 I$, $\mu = \mu_1 \boldsymbol{e}_r \otimes \boldsymbol{e}_r + \mu_2 I$, $\alpha = \kappa = \mathrm{i}\chi \boldsymbol{e}_r^\times$. For such a medium one obtains

$$P = -2\mathrm{i}k\chi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad Q = k^2(\varepsilon_2\mu_2 + \chi^2)\hat{1} \qquad \nu^2 = l(l+1)\frac{\varepsilon_2\mu_2}{\varepsilon_1\mu_1}. \tag{46}$$

The matrix solutions T_1 and T_2 equal

$$\begin{gathered}
T_1 = \begin{pmatrix} \exp(\mathrm{i}\chi k r) j_{s_1}(n_1 k r) & 0 \\ 0 & j_{s_1}(n_2 k r) \end{pmatrix} \qquad T_2 = \begin{pmatrix} \exp(\mathrm{i}\chi k r) j_{s_2}(n_1 k r) & 0 \\ 0 & j_{s_2}(n_2 k r) \end{pmatrix} \\
n_1 = \sqrt{\varepsilon_2\mu_2 + 2\chi^2} \qquad n_2 = \sqrt{\varepsilon_2\mu_2 + \chi^2}.
\end{gathered} \tag{47}$$

Tensors η_i and ζ_i $(i = 1, 2)$ are of the form

$$\begin{aligned}
\eta_i &= \mathrm{e}^{\mathrm{i}\chi k r} j_{s_i}(n_1 k r)\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta - \left(\frac{\mathrm{i}(r j_{s_i}(n_2 k r))'}{\mu_2 k r} - \frac{\chi}{\mu_2} j_{s_i}(n_2 k r)\right)\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\varphi \\
\zeta_i &= j_{s_i}(n_2 k r)\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\varphi + \mathrm{e}^{\mathrm{i}\chi k r}\frac{\mathrm{i}(r j_{s_i}(n_1 k r))'}{\varepsilon_2 k r}\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta .
\end{aligned} \tag{48}$$

The impedance tensors of the partial waves can be written as

$$\Gamma_i(r) = \frac{\mathrm{i}}{k\varepsilon_2}\frac{(r j_{s_i}(n_1 k r))'}{r j_{s_i}(n_1 k r)}\boldsymbol{e}_\varphi \otimes \boldsymbol{e}_\theta + \mathrm{i}k\mu_2\left(\frac{(r j_{s_i}(n_2 k r))'}{r j_{s_i}(n_2 k r)} + \mathrm{i}k\chi\right)^{-1}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\varphi . \tag{49}$$

5. Applications of the operator method

In this section we apply evolution and impedance operators introduced above to wave propagation in dielectric spherical guides and scattering by spherical particles.

5.1. Spherical waveguides

Let us consider a bianisotropic guide, which consists of three regions:

$$(\varepsilon, \mu, \alpha, \kappa) = \begin{cases} (\varepsilon_1, \mu_1, \alpha_1, \kappa_1) & \text{for } 0 < r < a \\ (\varepsilon_3, \mu_3, \alpha_3, \kappa_3) & \text{for } a < r < b \\ (\varepsilon_2, \mu_2, \alpha_2, \kappa_2) & \text{for } r > b. \end{cases} \tag{50}$$

We assume that the guiding of electromagnetic radiation can occur in the layer $a < r < b$ enclosed between two claddings. Such wave propagation takes place for atmosphere radio waves reflecting both from the earth and ionosphere.

In the central region $0 < r < a$ only waves, which are described by the spherical Bessel functions of the positive order, can propagate. Spherical Bessel functions of the negative order become infinite in the point $r = 0$, and this case cannot be realized in physics. For example, the solutions of the isotropic central region are functions j_l. In the spherical layer $a < r < b$, electromagnetic wave is characterized by the evolution operator Ω_a^r, which is expressed by means of the couple of spherical Bessel or Hankel functions. In the outer cladding extending endlessly the solution should correspond to the progressing wave originating from the surface $r = b$. Such waves are described by the spherical Hankel functions of the first kind $h^{(1)}$. Just as for the plane [10] and cylindrical [12] waveguides, the dispersion equation of the spherical guide is of the form

$$\mathrm{tr}(\overline{\Theta}) = 0 \qquad \Theta = (\widetilde{\Gamma}_{\mathrm{cl2}} \quad -I)\Omega_a^b \begin{pmatrix} I \\ \Gamma_{\mathrm{cl1}} \end{pmatrix}, \tag{51}$$

where $\overline{\Theta}$ is the adjoint tensor to the tensor Θ; $\Gamma_{\mathrm{cl1}} = \Gamma_{\mathrm{cl1}}(a)$, $\widetilde{\Gamma}_{\mathrm{cl2}} = \widetilde{\Gamma}_{\mathrm{cl2}}(b)$ are the surface impedance tensors of the central and outer claddings; Ω_a^b is the evolution operator of the guiding layer continued from the point $r = a$ to the point $r = b$. If the region $a < r < b$ contains n bianisotropic spherical layers, then the evolution operator Ω_a^b is equal to the product of the evolution operators of these layers:

$$\Omega_a^b = \Omega_{a_{n-1}}^b \cdots \Omega_{a_1}^{a_2} \Omega_a^{a_1}. \tag{52}$$

Usually a dispersion equation determines the dependence of the propagation constant (longitudinal wave number) on frequency. For spherical waveguides the dispersion equation (51) gives the frequency dependence of the wave number l. At $a \to \infty$ the transfer from the spherical waveguide to the plane one is possible. For such a plane waveguide the propagation constant is the linear wave number l/a, while the linear distance is $a\theta$. Electromagnetic modes in spherical waveguides are always leaky, i.e. the wave number is a complex number: $l = l' + \mathrm{i}l''$. The real and imaginary parts of the wave number describe wave propagation and attenuation, respectively.

Let us study the modes of the isotropic spherical guide of the form (50), the inner and outer claddings of which are characterized by the positive values of the refractive index, while the guiding layer possesses simultaneously negative values of the dielectric permittivity and magnetic permeability [18] in the frequency range $f = \omega/2\pi < 18.5$ GHz:

$$\varepsilon_3(\omega) = 1 + \frac{\omega_{pe}^2}{\omega_{1e}^2 - \omega^2 - \mathrm{i}\gamma_e\omega} \qquad \mu_3(\omega) = 1 + \frac{\omega_{pm}^2}{\omega_{1m}^2 - \omega^2 - \mathrm{i}\gamma_m\omega}, \tag{53}$$

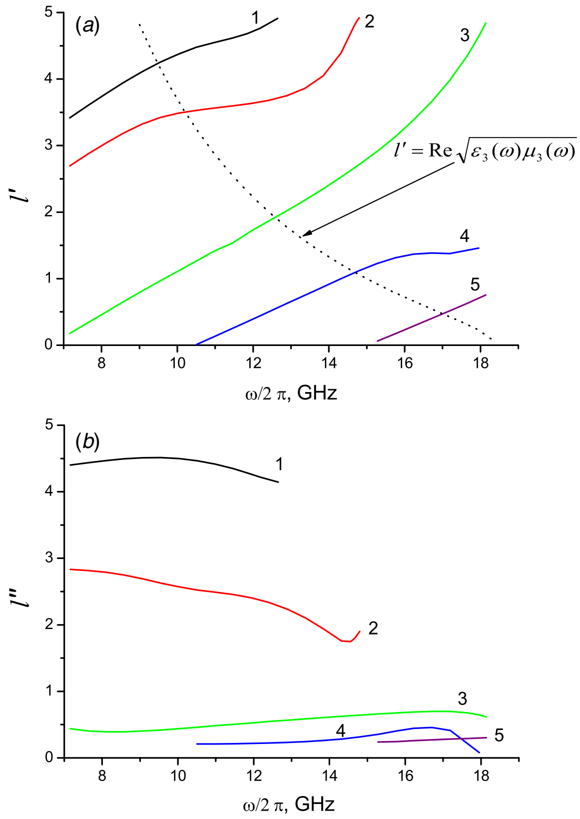

Figure 1. Frequency dependence of (a) real l' and (b) imaginary l'' parts of the wave number for the negative-refractive-index spherical waveguide. Dielectric permittivity and magnetic permeability of the metamaterial are described by formulae (53). Waveguide parameters: $\varepsilon_1 = 2$, $\mu_1 = 1$, $\varepsilon_2 = 1$, $\mu_2 = 1$, $a = 1$ cm, $b = 1.2$ cm.

where $\omega_{pe} = 1.1543 \times 10^{11}$ s^{-1}, $\omega_{pm} = 1.6324 \times 10^{11}$ s^{-1}, $\omega_{1e} = \omega_{1m} = 2\pi \times 5 \times 10^6$ s^{-1}, $\gamma_e = 2\gamma_m = 2\pi \times 6 \times 10^6$ s^{-1}. Such a frequency dispersion of the metamaterial (negative-refractive-index material, left-handed material) was utilized in paper [19].

Negative-refractive-index waveguides support fast and slow guided modes [20, 21]. Fast (slow) modes have the phase velocity greater (less) than that of the wave in a homogeneous medium. In figure 1(a) the fast and slow modes are separated by the dotted curve $l' = \mathrm{Re}\sqrt{\varepsilon_3(\omega)\mu_3(\omega)}$. Fast mode wave numbers l' almost linearly depend on frequency. Therefore, the group velocity of the fast waves $v_g \sim \partial\omega/\partial l'$ changes insignificantly right up to the frequencies marked by the dotted curve. The imaginary part l'' of the wave number is almost constant for fast modes.

Modes 1 and 2 possess the greatest attenuation (see figure 1(b)). The values of l' and l'' are approximately equal for such modes. In practical use the waves with small attenuation are preferred; they are modes 4 and 5. At a frequency near 17.5 GHz the imaginary parts of their wave numbers become equal. In this point the modes 4 and 5 are distinguishable only in phase velocity. Near the frequency 18 GHz the mode 4 has the smallest attenuation and can propagate at a large distance. If l' is small, then the wave number $l \approx \mathrm{i}l''$ and such modes are called emitting waves. Emitting modes do not propagate; they are purely damped (evanescent) waves. In figure 1(a) one can note the frequency regions, in which the modes 3, 4 and 5 are emitting modes.

5.2. Surface electromagnetic waves

Surface electromagnetic waves at the spherical interface are leaky modes as the waveguide modes considered in the previous subsection. Dispersion equation for surface waves can be obtained from equation (51), if the spherical layer between two claddings is absent ($a = b$). In this case $\Omega_a^a = E$ and from (51) it follows the dispersion equation for surface waves:

$$\mathrm{tr}(\overline{\widetilde{\Gamma}_{\mathrm{cl2}} - \Gamma_{\mathrm{cl1}}}) = 0. \tag{54}$$

Let us consider the surface electromagnetic waves at the spherical interface $r = a$ between the isotropic medium $\varepsilon_2 = \varepsilon_2 1, \mu_2 = \mu_2 1$ and the bianisotropic medium $\varepsilon_1 = \varepsilon_1 1, \mu_1 = \mu_1 1, \alpha_1 = \kappa_1 = \mathrm{i}\chi \boldsymbol{e}_r^{\times}$ (1 is the unit tensor in the three-dimensional space). By substituting the impedance tensors of each medium (45) and (49) to equation (54) we obtain two dispersion relations, the first of which is the equation for TE-polarized waves

$$\frac{1}{\varepsilon_1}\frac{(aj_l(n_1ka))'}{j_l(n_1ka)} = \frac{1}{\varepsilon_2}\frac{\left(ah_l^{(1)}\left(\sqrt{\varepsilon_2\mu_2}ka\right)\right)'}{h_l^{(1)}\left(\sqrt{\varepsilon_2\mu_2}ka\right)} \tag{55}$$

and the second equation describes TM-waves

$$\frac{1}{\mu_1}\left(\frac{(aj_l(n_2ka))'}{j_l(n_2ka)} + \mathrm{i}ka\chi\right) = \frac{1}{\mu_2}\frac{\left(ah_l^{(1)}\left(\sqrt{\varepsilon_2\mu_2}ka\right)\right)'}{h_l^{(1)}\left(\sqrt{\varepsilon_2\mu_2}ka\right)}, \tag{56}$$

where $n_1 = \sqrt{\varepsilon_1\mu_1 + 2\chi^2}, n_2 = \sqrt{\varepsilon_1\mu_1 + \chi^2}$, the prime denotes a-differentiation.

Dispersion curves in figure 2(a) express almost linear dependence of the real part of the wave number on the normalized frequency ka. Evidently, this linear dependence is formed owing to very weak distinction between isotropic and bianisotropic media. The slopes of all dispersion curves are approximately equal; therefore, the group velocities of these surface waves are very close. It should be noted that the curves $l'(ka)$ for TM modes are situated above the TE-mode curves, but the imaginary parts of wave numbers $l''(ka)$ for TE modes are greater. The larger cuts off the less attenuation of the mode. The lower TM mode has the smallest attenuation of the waves, the curves of which are shown in figures.

5.3. Scattering of electromagnetic waves by spherical particles

Let the electromagnetic wave, the field strengths of which are $\boldsymbol{H}^{(0)}$ and $\boldsymbol{E}^{(0)}$, is incident from vacuum ($\varepsilon = 1, \mu = 1$) on the n-layered spherical particle with bianisotropic constitutive parameters

$$(\varepsilon, \mu, \alpha, \kappa) = \begin{cases} (\varepsilon_1, \mu_1, \alpha_1, \kappa_1) & \text{for } 0 < r < a_1 \\ (\varepsilon_j, \mu_j, \alpha_j, \kappa_j) & \text{for } a_{j-1} < r < a_j, \quad j = 2, \ldots, n. \end{cases} \tag{57}$$

The incident wave induces the field inside the spherical particle, which values in the outer nth cladding can be written as

$$\begin{pmatrix} \boldsymbol{H}(r,\theta,\varphi) \\ \boldsymbol{E}(r,\theta,\varphi) \end{pmatrix} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \mathrm{e}^{\mathrm{i}m\varphi} F_l^m(\theta) V_l(r) \Omega_{a_1}^r[l] \begin{pmatrix} I \\ \Gamma_l \end{pmatrix} \boldsymbol{H}_{tl}(a_1), \tag{58}$$

where the tensor F_l^m is determined by formula (20), $V_l(r)$ is the matrix (24) restoring the total fields in the nth cladding using their tangential components and corresponding to the number l, $\Omega_{a_1}^r[l]$ is the evolution operator for the lth spherical wave, $\Gamma_l = \Gamma_l(a_1)$ is the wave impedance

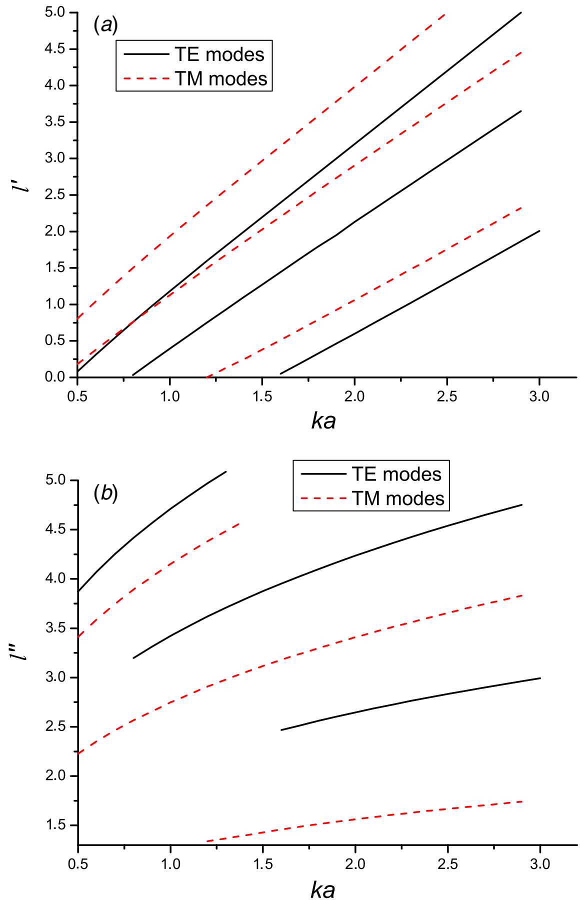

Figure 2. The solution of dispersion equations (55) and (56) for surface waves at the spherical interface between isotropic and bianisotropic media. Parameters of media: $\varepsilon_1 = \varepsilon_2 = 2.1$, $\mu_1 = \mu_2 = 1$, $\chi = 0.1$.

tensor in the first layer at the interface $r = a_1$. The quantities in (58) to be found are the tangential components of the magnetic field $\boldsymbol{H}_{tl}(a_1)$.

The scattered wave propagates in vacuum and is described in the same way as the field inside the spherical particle:

$$\begin{pmatrix} \boldsymbol{H}^{(sc)}(r,\theta,\varphi) \\ \boldsymbol{E}^{(sc)}(r,\theta,\varphi) \end{pmatrix} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \mathrm{e}^{\mathrm{i}m\varphi} F_l^m(\theta) V_l^{(sc)}(r) \begin{pmatrix} I \\ \widetilde{\Gamma}_l(r) \end{pmatrix} \widetilde{\eta}_{1l}(r)\widetilde{\eta}_{1l}^{-}(a_n)\boldsymbol{H}_{tl}^{(sc)}(a_n). \tag{59}$$

The quantities with tildes are expressed in terms of the spherical Hankel functions of the first kind $h_l^{(1)}$. During scattering we are interested in the fields $\boldsymbol{H}^{(sc)}$, $\boldsymbol{E}^{(sc)}$ in infinity. At $r \to \infty$ the spherical Hankel functions correspond to the divergent spherical wave $h_l^{(1)}(kr) \sim \mathrm{e}^{\mathrm{i}kr}/r$, then we obtain

$$\widetilde{\eta}_{1l}(r) \sim \frac{\mathrm{e}^{\mathrm{i}kr}}{r} I \qquad \widetilde{\Gamma}_l(r) \approx -\boldsymbol{e}_r^{\times} \qquad V_l^{(sc)}(r) \approx E \equiv \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

The impedance tensor does not depend neither on the number l nor on the radial coordinate r. Therefore, the scattered field can be noticeably simplified and rewritten as

$$\begin{pmatrix} \boldsymbol{H}^{(sc)}(r,\theta,\varphi) \\ \boldsymbol{E}^{(sc)}(r,\theta,\varphi) \end{pmatrix} = \frac{\mathrm{e}^{\mathrm{i}kr}}{r} \left(\frac{\mathrm{e}^{\mathrm{i}ka_n}}{a_n}\right)^{-1} \begin{pmatrix} I \\ -\boldsymbol{e}_r^{\times} \end{pmatrix} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \mathrm{e}^{\mathrm{i}m\varphi} F_l^m(\theta) \boldsymbol{H}_{tl}^{(sc)}(a_n). \tag{60}$$

Function of angles θ and φ near $\mathrm{e}^{\mathrm{i}kr}/r$ in equation (60) is called the scattering amplitude. Unknown fields $\boldsymbol{H}_{tl}(a_1)$ and $\boldsymbol{H}_{tl}^{(sc)}(a_n)$ can be determined from continuity conditions for the tangential components of electric and magnetic fields at the surface of the multi-layered particle $r = a_n$:

$$\begin{pmatrix} \boldsymbol{H}_t^{(0)}(a_n,\theta,\varphi) \\ \boldsymbol{E}_t^{(0)}(a_n,\theta,\varphi) \end{pmatrix} + \begin{pmatrix} \boldsymbol{H}_t^{(sc)}(a_n,\theta,\varphi) \\ \boldsymbol{E}_t^{(sc)}(a_n,\theta,\varphi) \end{pmatrix} = \begin{pmatrix} \boldsymbol{H}_t(a_n,\theta,\varphi) \\ \boldsymbol{E}_t(a_n,\theta,\varphi) \end{pmatrix}. \tag{61}$$

Designating $\boldsymbol{W}^{(0)} = \left(\boldsymbol{H}_t^{(0)}, \boldsymbol{E}_t^{(0)}\right)^T$ and substituting the scattering field strengths and fields inside the particle one can write the following expression as boundary conditions:

$$\begin{aligned} \boldsymbol{W}^{(0)}(a_n,\theta,\varphi) &+ \begin{pmatrix} I \\ -\boldsymbol{e}_r^{\times} \end{pmatrix} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \mathrm{e}^{\mathrm{i}m\varphi} F_l^m(\theta) \boldsymbol{H}_{tl}^{(sc)}(a_n) \\ &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \mathrm{e}^{\mathrm{i}m\varphi} F_l^m(\theta) \Omega_{a_1}^{a_n}[l] \begin{pmatrix} I \\ \Gamma_l \end{pmatrix} \boldsymbol{H}_{tl}(a_1). \end{aligned} \tag{62}$$

After multiplication of equation (62) by $\mathrm{e}^{-\mathrm{i}m'\varphi} \sin^2\theta P_{l'}^{|m'|}(\cos\theta)/2\pi$ and integration over φ from 0 to 2π and over θ from 0 to π we obtain

$$\begin{aligned} \boldsymbol{W}_{m,l}^{(0)}(a_n) &+ \begin{pmatrix} I \\ -\boldsymbol{e}_r^{\times} \end{pmatrix} \left(\mathrm{i}m\boldsymbol{H}_{tl}^{(sc)}(a_n) + \boldsymbol{e}_r^{\times} u_l^{(-1)} \boldsymbol{H}_{tl-1}^{(sc)}(a_n) + \boldsymbol{e}_r^{\times} u_l^{(1)} \boldsymbol{H}_{tl+1}^{(sc)}(a_n)\right) \\ &= \mathrm{i}m\Omega_{a_1}^{a_n}[l] \begin{pmatrix} I \\ \Gamma_l \end{pmatrix} \boldsymbol{H}_{tl}(a_1) + \boldsymbol{e}_r^{\times} u_l^{(-1)} \Omega_{a_1}^{a_n}[l-1] \begin{pmatrix} I \\ \Gamma_{l-1} \end{pmatrix} \boldsymbol{H}_{tl-1}(a_1) \\ &\quad + \boldsymbol{e}_r^{\times} u_l^{(1)} \Omega_{a_1}^{a_n}[l+1] \begin{pmatrix} I \\ \Gamma_{l+1} \end{pmatrix} \boldsymbol{H}_{tl+1}(a_1), \end{aligned} \tag{63}$$

where

$$\begin{aligned} &\boldsymbol{W}_{m,l}^{(0)}(a_n) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\varphi\, \mathrm{e}^{-\mathrm{i}m\varphi} \int_0^{\pi} \mathrm{d}\theta \sin^2\theta P_l^{|m|}(\cos\theta), \qquad \boldsymbol{W}^{(0)}(a_n,\theta,\varphi) \\ &u_l^{(-1)} = -\frac{l(l-|m|+1)(l+|m|+1)!}{(2l+1)(l+3/2)(l-|m|+1)!} \qquad u_l^{(1)} = \frac{(l+|m|)(l+1)(l+|m|-1)!}{(2l+1)(l-1/2)(l-|m|-1)!}. \end{aligned} \tag{64}$$

First, the magnetic field strengths $\boldsymbol{H}_{tl}(a_1)$ should be found. For that one should multiply equation (63) by the block row matrix $\left(\boldsymbol{e}_r^{\times}\ \ I\right)$. Then all scattered field amplitudes $\boldsymbol{H}_{tl}^{(sc)}(a_n)$ disappear and the following recurring relation can be written for $l > |m|$:

$$\boldsymbol{H}_{tl+1} = A_l \boldsymbol{H}_{tl} + B_l \boldsymbol{H}_{tl-1} + \boldsymbol{C}_l, \tag{65}$$

where

$$\begin{aligned} A_l &= -\frac{\mathrm{i}m}{u_l^{(1)}} \left[\left(-I\ \ \boldsymbol{e}_r^{\times}\right) \Omega_{a_1}^{a_n}[l+1] \begin{pmatrix} I \\ \Gamma_{l+1} \end{pmatrix}\right]^{-} \left(\boldsymbol{e}_r^{\times}\ \ I\right) \Omega_{a_1}^{a_n}[l] \begin{pmatrix} I \\ \Gamma_l \end{pmatrix} \\ B_l &= -\frac{u_l^{(-1)}}{u_l^{(1)}} \left[\left(-I\ \ \boldsymbol{e}_r^{\times}\right) \Omega_{a_1}^{a_n}[l+1] \begin{pmatrix} I \\ \Gamma_{l+1} \end{pmatrix}\right]^{-} \left(-I\ \ \boldsymbol{e}_r^{\times}\right) \Omega_{a_1}^{a_n}[l-1] \begin{pmatrix} I \\ \Gamma_{l-1} \end{pmatrix} \\ \boldsymbol{C}_l &= \frac{1}{u_l^{(1)}} \left[\left(-I\ \ \boldsymbol{e}_r^{\times}\right) \Omega_{a_1}^{a_n}[l+1] \begin{pmatrix} I \\ \Gamma_{l+1} \end{pmatrix}\right]^{-} \left(\boldsymbol{e}_r^{\times}\ \ I\right) \boldsymbol{W}_{m,l}^{(0)}(a_n). \end{aligned} \tag{66}$$

Number l can take the minimal value $l = |m|$. Equation (65) is not satisfied for $l = |m|$, because $u_l^{(1)}$ becomes zero. At $l = |m|$ from the boundary conditions it follows the equation

$$u_l^{(1)} A_{|m|} \boldsymbol{H}_{t|m|} + u_l^{(1)} \boldsymbol{C}_{|m|} = 0 \tag{67}$$

from which the mth magnetic field amplitude can be easily found

$$\boldsymbol{H}_{t|m|} = -\left(u_l^{(1)} A_{|m|}\right)^{-}\left(u_l^{(1)} \boldsymbol{C}_{|m|}\right). \tag{68}$$

Recurring equations (65) can be rewritten by means of one vector $\boldsymbol{H}_{t|m|+1}$ as

$$\boldsymbol{H}_{t|m|+q+1} = \boldsymbol{h}_q + G_q \boldsymbol{H}_{t|m|+1} \qquad q = 1, 2, \ldots \tag{69}$$

where

$$\begin{aligned} \boldsymbol{h}_1 &= \boldsymbol{C}_{|m|+1} - B_{|m|+1}\left(u_l^{(1)} A_{|m|}\right)^{-}\left(u_l^{(1)} \boldsymbol{C}_{|m|}\right) \qquad & G_1 &= A_{|m|+1} \\ \boldsymbol{h}_q &= \boldsymbol{C}_{|m|+q} + A_{|m|+q} \boldsymbol{h}_{q-1} & G_q &= B_{|m|+q} + A_{|m|+q} G_{q-1}. \end{aligned} \tag{70}$$

Vectors $\boldsymbol{h}_q$ and tensors G_q can be calculated for any integer number q. Let us assume that we have computed the magnetic fields until $\boldsymbol{H}_{t|m|+N+1}$, where $N \gg 1$. The values of amplitudes $\boldsymbol{H}_{tl}$ should decrease, if l increases, in order that solution written as the l-series converges. That is why there is a great number N, for which the magnetic field becomes zero

$$\boldsymbol{H}_{t|m|+N+1} = 0. \tag{71}$$

From this equation it is easy to find $\boldsymbol{H}_{t|m|+1}$:

$$\boldsymbol{H}_{t|m|+1} = -G_N^{-} \boldsymbol{h}_N. \tag{72}$$

The field amplitudes inside the spherical particles are calculated according to formulae (69):

$$\boldsymbol{H}_{t|m|+q+1} = \boldsymbol{h}_q - G_q G_N^{-} \boldsymbol{h}_N \qquad q = 1, \ldots, N-1. \tag{73}$$

To determine the scattered magnetic field amplitudes $\boldsymbol{H}_{tl}^{(sc)}$ we multiply the boundary conditions (63) by the block row matrix $\begin{pmatrix} 0 & I \end{pmatrix}$. Recurring relation for the scattered fields takes the form

$$\boldsymbol{H}_{tl+1}^{(sc)} = A_l^{(sc)} \boldsymbol{H}_{tl}^{(sc)} + B_l^{(sc)} \boldsymbol{H}_{tl-1}^{(sc)} + \boldsymbol{C}_l^{(sc)}, \tag{74}$$

where

$$A_l^{(sc)} = \frac{\mathrm{i}m}{u_l^{(1)}} \boldsymbol{e}_r^{\times} \qquad B_l^{(sc)} = -\frac{u_l^{(-1)}}{u_l^{(1)}} I$$

$$\begin{aligned} \boldsymbol{C}_l^{(sc)} = \frac{1}{u_l^{(1)}} \Bigg[& \mathrm{i}m \begin{pmatrix} 0 & I \end{pmatrix} \Omega_{a_1}^{a_n}[l] \begin{pmatrix} I \\ \Gamma_l \end{pmatrix} \boldsymbol{H}_l + u_l^{(-1)} \begin{pmatrix} 0 & \boldsymbol{e}_r^{\times} \end{pmatrix} \Omega_{a_1}^{a_n}[l-1] \begin{pmatrix} I \\ \Gamma_{l-1} \end{pmatrix} \boldsymbol{H}_{l-1} \\ & + u_l^{(1)} \begin{pmatrix} 0 & \boldsymbol{e}_r^{\times} \end{pmatrix} \Omega_{a_1}^{a_n}[l+1] \begin{pmatrix} I \\ \Gamma_{l+1} \end{pmatrix} \boldsymbol{H}_{l+1} - \begin{pmatrix} 0 & I \end{pmatrix} \boldsymbol{W}_{m,l}^{(0)}(a_n) \Bigg]. \end{aligned} \tag{75}$$

Equation (74) can be solved in the same way as equation (65). In formulae (67)–(73) one should replace A_l, B_l, $\boldsymbol{C}_l$ by $A_l^{(sc)}$, $B_l^{(sc)}$, $\boldsymbol{C}_l^{(sc)}$, respectively. In such a way we can find scattered magnetic fields $\boldsymbol{H}_{tl}^{(sc)}$, where $l = |m|, \ldots, |m| + N$. The scattered field is characterized by the differential scattering cross-section (the power radiated in the direction $\boldsymbol{e}_r$ per unit solid angle)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}o} = r^2 \frac{|\boldsymbol{H}^{(sc)}|^2}{|\boldsymbol{H}^{(0)}|^2}. \tag{76}$$

By substituting the expression for the scattered magnetic field (60) into the formula for differential cross-section one obtains

$$\frac{d\sigma}{do} = \frac{a_n^2}{|\boldsymbol{H}^{(0)}|^2} \left| \sum_{l=0}^{\infty} \sum_{m=-l}^{l} e^{im\varphi} F_l^m(\theta) \boldsymbol{H}_{tl}^{(sc)}(a_n) \right|^2 . \tag{77}$$

Polarization averaging (angle φ averaging) gives the equation

$$\frac{d\sigma}{do} = \frac{a_n^2}{|\boldsymbol{H}^{(0)}|^2} \sum_{m=-\infty}^{\infty} \left| \sum_{l=|m|}^{\infty} F_l^m(\theta) \boldsymbol{H}_{tl}^{(sc)}(a_n) \right|^2 , \tag{78}$$

where θ is the scattering angle. In the sum over l only first N amplitudes $\boldsymbol{H}_{tl}^{(sc)}$ are nonzero.

So, one should execute the following steps to solve the scattering problem.

(i) Choose the incident wave and parameters of the multi-layer dielectric spherical particle.
(ii) Find the tensors η and ζ (35) for each spherical layer. Calculate evolution operators (36) and impedance tensors (38) of the layers.
(iii) Determine the amplitudes of initial fields $\boldsymbol{W}_{m,l}^{(0)}$ using formula (64).
(iv) Compute matrices A_l, B_l and vectors $\boldsymbol{C}_l$ by substituting evolution operators and impedance tensors into equations (66).
(v) Determine the values of the vectors $\boldsymbol{h}_q$ and matrices G_q from the recurring relations (70).
(vi) Find the magnetic field amplitudes at the surface of the spherical particle $\boldsymbol{H}_{tl}$ from $l = |m|$ to $l = |m| + N$ (formulae (68), (73)).
(vii) Substitute the fields $\boldsymbol{H}_{tl}$ into (75). Carry out the items (v) and (vi) to solve the recurring equations (74) for the scattered fields.
(viii) Compute the differential cross-section (78) using the fields $\boldsymbol{H}_{tl}^{(sc)}$.

As an example we consider the incidence of the x-polarized plane electromagnetic wave with unit amplitude $|\boldsymbol{H}^{(0)}|^2 = 1$

$$\begin{pmatrix} \boldsymbol{H}^{(0)}(r,\theta,\varphi) \\ \boldsymbol{E}^{(0)}(r,\theta,\varphi) \end{pmatrix} = e^{ikr\cos\theta} \begin{pmatrix} \cos\varphi\sin\theta\boldsymbol{e}_r + \sin\varphi\cos\theta\boldsymbol{e}_\theta + \cos\varphi\boldsymbol{e}_\varphi \\ -\sin\varphi\sin\theta\boldsymbol{e}_r - \cos\varphi\cos\theta\boldsymbol{e}_\theta + \sin\varphi\boldsymbol{e}_\varphi \end{pmatrix} \tag{79}$$

onto the isotropic dielectric particle

$$(\varepsilon,\mu) = \begin{cases} (\varepsilon_1,\mu_1) & \text{for } 0 < r < a_1 \\ (\varepsilon_2,\mu_2) & \text{for } a_1 < r < a_2. \end{cases} \tag{80}$$

For the incident field (79) only two values of the number m are possible: $m = -1$ and $m = 1$. Therefore, the differential cross-section equals

$$\frac{d\sigma}{do} = a_2^2 \left(\left| \sum_{l=1}^{\infty} F_l^1(\theta) \boldsymbol{H}_{tl}^{(sc)}(a_2, m=1) \right|^2 + \left| \sum_{l=1}^{\infty} F_l^{-1}(\theta) \boldsymbol{H}_{tl}^{(sc)}(a_2, m=-1) \right|^2 \right). \tag{81}$$

In figures 3 and 4 the results of calculation of differential cross-section for the light scattered by the spherical particles (80) are presented. We computed magnetic fields $\boldsymbol{H}_{t|m|+q}^{(sc)}$ until $q = N = 15$.

For lower frequencies of the incident light (dashed line in figure 3) the scattering occurs approximately evenly in all scattering directions. The scattering maximum falls on the angle $\theta = 0^\circ$. At greater frequencies the scattering becomes nonuniform. Differential cross-section takes maximal values for the angles $\theta = 0^\circ; 180^\circ$, while minimal scattering occurs at 90°. Such angle dependence can be explained as follows. At normal incidence electromagnetic

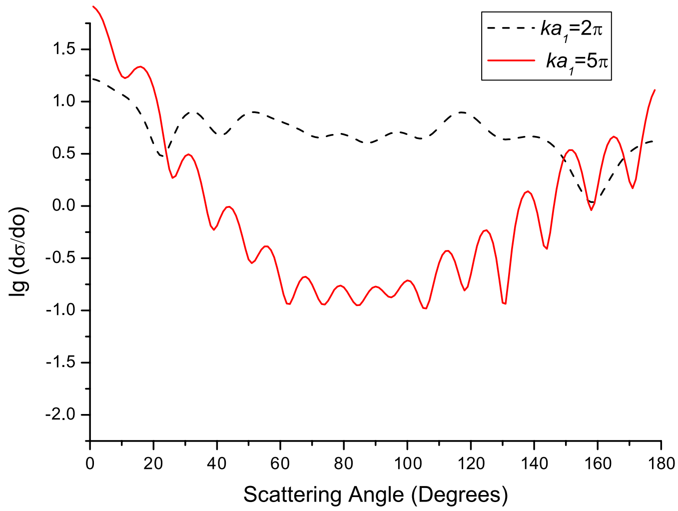

Figure 3. Differential cross-section versus scattering angle for two different frequencies. Parameters: $\varepsilon_1 = 2 + 0.2\mathrm{i}$, $\mu_1 = 1$, $\varepsilon_2 = 3 + 0.1\mathrm{i}$, $\mu_2 = 1.2$, $a_2 = 1.3a_1$.

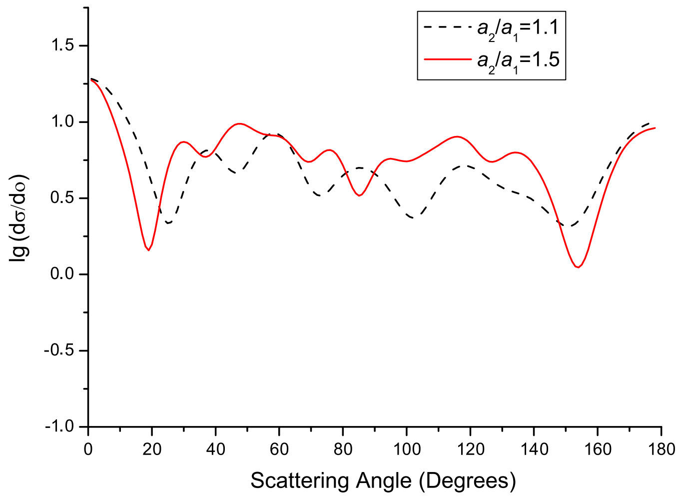

Figure 4. Differential cross-section versus scattering angle for two different thicknesses a_2. Parameters: $\varepsilon_1 = 2 + 0.2\mathrm{i}$, $\mu_1 = 1$, $\varepsilon_2 = 3 + 0.1\mathrm{i}$, $\mu_2 = 1.2$, $ka_1 = 2\pi$.

waves of large frequency strongly reflect from the spherical particle. For other angles of incidence the greater part of the power passes through the particle. That is why the forward and backward scattering are maximal. The increase of the outer layer radius does not lead to the essential change of the scattering pattern (see figure 4). One may note the deepening of the scattering minima near the angles $\theta = 20^\circ$ and $\theta = 160^\circ$, as well as the invariability of the forward ($\theta = 0^\circ$) and backward ($\theta = 180^\circ$) scattering.

6. Conclusion

Separation of variables r, θ and φ in the field strengths plays an important part. Owing to separation we can formulate the operator approach and use the same formulae for multi-layer structures with plane, cylindrical or spherical symmetry. For example, both for plane, cylindrical and spherical waveguides the dispersion equation (51) is the same. One should only substitute the appropriate evolution operators and surface impedance tensors of the layers. The separation of variables contributes also to formulating a more clear algorithm to determine the scattering fields than that using the method of spherical vector wavefunctions. As a matter of fact, the scattering problem is reduced to the execution of addition and multiplication of 2×2 and 4×4 matrices, and such operations do not take much time to calculate.

The operator method allows us to describe surface waves and waveguide modes applied in optical electronics and in the study of electromagnetic wave propagation in the Earth atmosphere. The light scattering by the dielectric spherical particles can be utilized in antenna and satellite communication system designs. In further publications we assume to investigate vector electromagnetic beams on the basis of general spherically symmetric solutions.

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